Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 45, Number 1, 2019, Pages 155–162

QUADRATURE FORMULAS FOR SIMPLE AND DOUBLE LAYER LOGARITHMIC POTENTIALS

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Abstract. Quadrature formulas for logarithmic potentials and normal derivative of simple layer logarithmic potential are constructed. Error estimations for the constructed quadrature formulas are given.

1. Introduction

One of the methods for solving boundary value problems for Laplace equation is a method of boundary integral equations (see [2]). In many cases it is impossible to find exact solution of integral equations of boundary value problems for Laplace equation, therefore there arises a need to have justification of collocation method for these integral equations, and for this one should first construct quadrature formulas for simple and double layer logarithmic potentials. Note that in [3], a cubature formula for simple layer acoustic potential was constructed, and in [4], a cubature formula for double layer acoustic potential was constructed. Then, based on these cubature formulas, the justification of collocation method for integral equations of boundary value problems for the Helmholtz equation in three-dimensional space was presented in [1, 5, 6, 7]. However, quadrature formulas for simple and double layer logarithmic potentials have not been constructed yet. This work is dedicated to this problem.

2. Dividing the curve into “regular” elementary parts

Suppose a simple closed Lyapunov curve $L \subset R^2$ is given by the parametric equation $x\,(t) = (x_1\,(t)\ , x_2\,(t))\ ,\ t \in [a, b]$. It is known (see [8]) that

$$m_1 = \min_{t\in[a,b]} \sqrt{\left(x_1'\,(t)\right)^2 + \left(x_2'\,(t)\right)^2} > 0$$

and

$$M_1 = \max_{t\in[a,b]} \sqrt{\left(x_1'\,(t)\right)^2 + \left(x_2'\,(t)\right)^2} < +\infty.$$

Let’s divide the interval $[a, b]$ into $n > 2M_1\,(b - a)\,/d$ equal parts:

2010 *Mathematics Subject Classification.* 45E05; 31B10.
Key words and phrases. quadrature formulas, logarithmic potentials, curvilinear integral, Lyapunov curve.

$t_k = a + \frac{(b-a)\,k}{n}$, $k = \overline{0,n}$, where d is a radius of a standard circle (see [9]). As control points we consider $x(\tau_k)$, where $\tau_k = a + \frac{(b-a)\,(2k-1)}{2n}$, $k = \overline{1,n}$. Let

$$L_k = \{\, x(t): \ \ t_{k-1} \le t \le t_k \}\,,$$
$$L'_k = \{\, x(t): \ \ t_{k-1} \le t \le \tau_k \}$$

and

$$L''_k = \{\, x(t) :: \tau_k \le t \le t_k \}\,, k = \overline{1,n}.$$

Then, using the arc length formula, we obtain

$$\frac{m_1\,(b-a)}{n} \le mesL_k \le \frac{M_1\,(b-a)}{n},$$
$$\frac{m_1\,(b-a)}{2n} \le mesL'_k \le \frac{M_1\,(b-a)}{2n}$$

and

$$\frac{m_1\,(b-a)}{2n} \le mesL''_k \le \frac{M_1\,(b-a)}{2n},$$

where $mesL$ denotes the length of the curve L. Besides, taking into account the inequalities (see [8])

$$M_0\, mesL'_k \le |x(\tau_k) - x(t_{k-1})| \le mesL'_k$$

and

$$M_0\, mesL''_k \le |x(t_k) - x(\tau_k)| \le mesL''_k,$$

where $M_0 \in (0,1)$ denotes a constant that does not depend on k and n, we obtain

(1) $\forall k \in \{1, 2, ..., n\} : r_k(n) \sim R_k(n)$, where

$$r_k\,(n) = \min\left\{|x(\tau_k) - x(t_{k-1})|\, |x(t_k) - x(\tau_k)|\right\}$$

and

$$R_k\,(n) = \max\left\{|x(\tau_k) - x(t_{k-1})|\ ,\ |x(t_k) - x(\tau_k)|\right\};$$

(2) $\forall k \in \{1, 2, ..., n\} : R_k\,(n) < d/2;$

(3) $\forall k, j \in \{1, 2, ..., n\} : r_j\,(n) \sim r_k\,(n)\,.$

In (1), $a\,(n) \sim b\,(n) \Leftrightarrow C_1 \le \frac{a(n)}{b(n)} \le C_2$ where C_1 and C_2 are positive constants that do not depend on n. It is clear that $r\,(n) \sim R\,(n) \sim \frac{1}{n}$, where $R\,(n) = \max\limits_{k=\overline{1,n}} R_k\,(n)$ and

$r\,(n) = \min\limits_{k=\overline{1,n}} r_k\,(n)$. Besides, it is not difficult to prove the following lemma.

Lemma 2.1. *There exist the constants $C'_0 > 0$ and $C'_1 > 0$ independent of n such that $\forall k, j \in \{1,\, 2, ...,\, n\}\,,\ j \ne k$ and $\forall y \in L_j$ the following inequality holds:*

$$C'_0\ |y - x(\tau_k)| \le |x(\tau_j) - x(\tau_k)| \le C'_1\ |y - x(\tau_k)|\,.$$

In the sequel, we will call this kind of division a division of the curve L into "regular" elementary parts.

3. Quadrature formula for simple layer logarithmic potential

Consider simple layer logarithmic potential

$$P(x) = \int_L \Phi(x,\, y)\,\rho(y) dL_y, \quad x = (x_1,\, x_2) \in L, \tag{3.1}$$

where $L \subset R^2$ is a simple closed Lyapunov curve with the index $0 < \alpha \leq 1$, $\rho(y)$ is a continuous function on the curve L, $\Phi(x, y)$ is a fundamental solution of the Laplace equation $\Delta u = 0$, i.e.

$$\Phi(x,\, y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|}, x, y \in R^2, x \neq y,$$

and Δ is a Laplace operator.

Denote by $C\,(L)$ a space of all continuous functions on L with the norm $\|\rho\|_\infty = \max\limits_{x\in L} |\rho\,(x)\,|$, and introduce a modulus of continuity of the form

$$\omega(\varphi,\, \delta) = \delta \sup_{\tau \geq \delta} \frac{\bar{\omega}(\varphi,\, \tau)}{\tau}, \quad \delta > 0,$$

for the function $\varphi(x) \in C\,(L)$, where $\bar{\omega}(\varphi,\, \tau) = \max\limits_{\substack{|x-y|\leq\tau \\ x,y\in L}} |\varphi(x) - \varphi(y)|$.

Divide the curve L into "regular" elementary parts $L = \bigcup\limits_{k=1}^{n} L_k$.

Theorem 3.1. *The expression*

$$P_n\,(x\,(\tau_k)) = \frac{b-a}{n} \sum_{\substack{j=1 \\ j\neq k}}^{n} \Phi\,(x\,(\tau_k)\,,\, x\,(\tau_j))\, \sqrt{(x_1'\,(\tau_j))^2 + (x_2'\,(\tau_j))^2}\, \rho\,(x\,(\tau_j))$$

is a quadrature formula for the integral (3.1) *at the control points* $x\,(\tau_k)$, $k = \overline{1,\, n}$, *and the following estimates are true:*

$\max\limits_{k=\overline{1,\,n}} |P\,(x\,(\tau_k)) - P_n\,(x\,(\tau_k))| \leq M \left(\omega\,(\rho,\, 1/n) + \|\rho\|_\infty \frac{1}{n^\alpha}\right)$ *if* $0 < \alpha < 1$,

$\max\limits_{k=\overline{1,\,n}} |P\,(x\,(\tau_k)) - P_n\,(x\,(\tau_k))| \leq M \left(\omega\,(\rho,\, 1/n) + \|\rho\|_\infty \frac{\ln n}{n}\right)$ *if* $\alpha = 1$.

Hereinafter M denotes a positive constant which can be different in different inequalities.

Proof. It is not difficult to see that

$$P\,(x\,(\tau_k)) - P_n\,(x\,(\tau_k)) = \int_{L_k} \Phi\,(x\,(\tau_k)\,,\, y)\;\rho\,(y)\, dL_y +$$

$$+ \sum_{\substack{j=1 \\ j\neq k}}^{n} \int_{L_j} (\Phi\,(x\,(\tau_k)\,,\, y) - \Phi\,(x\,(\tau_k)\,,\, x\,(\tau_j)))\;\rho\,(y)\, dL_y +$$

$$+ \sum_{\substack{j=1 \\ j\neq k}}^{n} \int_{L_j} \Phi\,(x\,(\tau_k)\,,\, x\,(\tau_j))\;(\rho\,(y) - \rho\,(x\,(\tau_j)))\, dL_y +$$

$$+\sum_{\substack{j=1\\ j\neq k}}^{n}\int\limits_{t_{j-1}}^{t_j}\Phi\left(x\left(\tau_k\right),\, x\left(\tau_j\right)\right)\left(\sqrt{\left(x_1'\left(t\right)\right)^2+\left(x_2'\left(t\right)\right)^2}-\right.$$

$$\left.-\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\right)\ \rho\left(x\left(\tau_j\right)\right)dt.$$

Denote the terms on the right-hand side of the last equality by $h_1^n\left(x\left(\tau_k\right)\right)$, $h_2^n\left(x\left(\tau_k\right)\right)$, $h_3^n\left(x\left(\tau_k\right)\right)$ and $h_4^n\left(x\left(\tau_k\right)\right)$, respectively.

Using the calculation formula for curvilinear integral, we find

$$\left|h_1^n\left(x\left(\tau_k\right)\right)\right|\leq\|\rho\|_\infty\int\limits_{L_k}\left|\Phi\left(x\left(\tau_k\right),\, y\right)\right|dL_y\leq$$

$$\leq M\ \|\rho\|_\infty\int\limits_0^{R(n)}\ln\left(\frac{1}{\tau}\right)d\tau\leq M\ \|\rho\|_\infty R\left(n\right)\ \left|\ln R\left(n\right)\right|.$$

Let $y\in L_j$ and $j\neq k$. Then, by Lemma 2.1, we have

$$\left|\Phi\left(x\left(\tau_k\right),\, y\right)-\Phi\left(x\left(\tau_k\right),\, x\left(\tau_j\right)\right)\right|=\frac{1}{2\pi}\left|\ln\left(1+\frac{\left|x\left(\tau_k\right)-x\left(\tau_j\right)\right|-\left|x\left(\tau_k\right)-y\right|}{\left|x\left(\tau_k\right)-y\right|}\right)\right|\leq$$

$$\leq\frac{1}{2\pi}\left|\ln\left(1+\frac{\left|x\left(\tau_j\right)-y\right|}{\left|x\left(\tau_k\right)-y\right|}\right)\right|\leq M\frac{R\left(n\right)}{\left|x\left(\tau_k\right)-y\right|}.$$

Consequently,

$$\left|h_2^n\left(x\left(\tau_k\right)\right)\right|\leq M\ \|\rho\|_\infty R\left(n\right)\int\limits_{r(n)}^{diam\,L}\frac{d\tau}{\tau}\leq M\ \|\rho\|_\infty R\left(n\right)\ \left|\ln R\left(n\right)\right|.$$

As the integral

$$\int\limits_L\left|\Phi\left(x,\, y\right)\right|dL_y$$

converges as an improper integral and

$$\int\limits_L\left|\Phi\left(x,\, y\right)\right|dL_y\leq M,\ \forall x\in L, \tag{3.2}$$

in view of Lemma 2.1 we obtain

$$\left|h_3^n\left(x\left(\tau_k\right)\right)\right|\leq M\,\omega\left(\rho, R\left(n\right)\right)\ \int_L\left|\Phi\left(x\left(\tau_k\right),\, y\right)\right|dL_y\leq M\omega\ \left(\rho, R\left(n\right)\right).$$

Obviously,

$$\begin{aligned}&\left|\sqrt{\left(x_1'\left(t\right)\right)^2+\left(x_2'\left(t\right)\right)^2}-\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\right|\leq\\ &\leq M\left(R\left(n\right)\right)^\alpha,\forall t\in\left[t_{j-1},t_j\right].\end{aligned} \tag{3.3}$$

Then, taking into account the inequality (3.2) and Lemma 2.1, we find

$$\left|h_4^n\left(x\left(\tau_k\right)\right)\right| \leq M \left\|\rho\right\|_\infty \left(R\left(n\right)\right)^\alpha \sum_{\substack{j=1\\ j\neq k}}^{n} \int\limits_{t_{j-1}}^{t_j} \left|\Phi\left(x\left(\tau_k\right),\, x\left(\tau_j\right)\right)\right| \, dt \leq$$

$$\leq M \left\|\rho\right\|_\infty \left(R\left(n\right)\right)^\alpha \sum_{\substack{j=1\\ j\neq k}}^{n} \int\limits_{L_j} \left|\Phi\left(x\left(\tau_k\right),\, y\right)\right| dL_y \leq$$

$$\leq M \left\|\rho\right\|_\infty \left(R\left(n\right)\right)^\alpha \int_L \left|\Phi\left(x\left(\tau_k\right),\, y\right)\right| dL_y \leq M \left\|\rho\right\|_\infty \left(R\left(n\right)\right)^\alpha .$$

As a result, summing up the estimates obtained above for the expressions $h_1^n\left(x\left(\tau_k\right)\right)$, $h_2^n\left(x\left(\tau_k\right)\right)$, $h_3^n\left(x\left(\tau_k\right)\right)$ and $h_4^n\left(x\left(\tau_k\right)\right)$, and taking into account the relation $R\left(n\right) \sim \frac{1}{n}$, we get the validity of Theorem 3.1. □

4. Quadrature formula for double layer logarithmic potential

Now let's construct the quadrature formula for double layer logarithmic potential

$$W(x) = \int\limits_L \frac{\partial\Phi(x,\, y)}{\partial\vec{n}(y)}\, \rho(y) dL_y, \quad x \in L, \tag{4.1}$$

where $L \subset R^2$ is a simple closed Lyapunov curve with the index $0 < \alpha \leq 1$, $\vec{n}(y)$ is an outer unit normal at the point $y \in L$, $\rho(y)$ is a continuous function on the curve L, and $\Phi(x, y)$ is a fundamental solution of the Laplace equation. Divide the curve L into "regular" elementary parts $L = \bigcup\limits_{k=1}^{n} L_k$.

The following theorem is true.

Theorem 4.1. *The expression*

$$W_n\left(x\left(\tau_k\right)\right) = \frac{b-a}{n} \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{\partial\Phi\left(x\left(\tau_k\right),\, x\left(\tau_j\right)\right)}{\partial\vec{n}\left(x\left(\tau_j\right)\right)} \sqrt{\left(x_1'\left(\tau_j\right)\right)^2 + \left(x_2'\left(\tau_j\right)\right)^2}\, \rho\left(x\left(\tau_j\right)\right)$$

is a quadrature formula for the integral (4.1) *at the control points* $x\left(\tau_k\right)$, $k = \overline{1,\, n}$, *and the following estimate is true:*

$$\max_{k=\overline{1,\, n}} \left|W\left(x\left(\tau_k\right)\right) - W_n\left(x\left(\tau_k\right)\right)\right| \leq M \left(\omega\left(\rho,\, 1/n\right) + \left\|\rho\right\|_\infty \frac{\ln n}{n^\alpha}\right).$$

Proof. It is not difficult to see that

$$W\left(x\left(\tau_k\right)\right) - W_n\left(x\left(\tau_k\right)\right) = \int\limits_{L_k} \frac{\partial\Phi\left(x\left(\tau_k\right),\, y\right)}{\partial\vec{n}\left(y\right)}\, \rho\left(y\right) dL_y +$$

$$+ \sum_{\substack{j=1\\ j\neq k}}^{n} \int\limits_{L_j} \frac{\partial\Phi\left(x\left(\tau_k\right),\, x\left(\tau_j\right)\right)}{\partial\vec{n}\left(x\left(\tau_j\right)\right)} \left(\rho\left(y\right) - \rho\left(x\left(\tau_j\right)\right)\right) dL_y +$$

$$+\sum_{\substack{j=1\\ j\neq k}}^{n}\int\limits_{L_j}\left(\frac{\partial\Phi\left(x\left(\tau_k\right),\,y\right)}{\partial\vec{n}\left(y\right)}-\frac{\partial\Phi\left(x\left(\tau_k\right),\,x\left(\tau_j\right)\right)}{\partial\vec{n}\left(x\left(\tau_j\right)\right)}\right)\rho\left(y\right)dL_y+$$

$$+\sum_{\substack{j=1\\ j\neq k}}^{n}\int\limits_{t_{j-1}}^{t_j}\frac{\partial\Phi\left(x\left(\tau_k\right),\,x\left(\tau_j\right)\right)}{\partial\vec{n}\left(x\left(\tau_j\right)\right)}\times$$

$$\times\left(\sqrt{\left(x_1'\left(t\right)\right)^2+\left(x_2'\left(t\right)\right)^2}-\sqrt{\left(x_1'\left(\tau_j\right)\right)^2+\left(x_2'\left(\tau_j\right)\right)^2}\right)\rho\left(x\left(\tau_j\right)\right)dt.$$

Denote the terms on the right-hand side of the last equality by $\delta_1^n\left(x\left(\tau_k\right)\right)$, $\delta_2^n\left(x\left(\tau_k\right)\right)$, $\delta_3^n\left(x\left(\tau_k\right)\right)$ and $\delta_4^n\left(x\left(\tau_k\right)\right)$, respectively.

It is known that

$$\left|\frac{\partial\Phi\left(x\left(\tau_k\right),\,y\right)}{\partial\vec{n}\left(y\right)}\right|=\frac{1}{2\pi}\left|\frac{\left(\overrightarrow{yx\left(\tau_k\right)},\vec{n}\left(y\right)\right)}{\left|x\left(\tau_k\right)-y\right|^2}\right|\leq\frac{M}{\left|x\left(\tau_k\right)-y\right|^{1-\alpha}}. \tag{4.2}$$

Then, by the calculation formula for curvilinear integral, we obtain

$$\left|\delta_1^n\left(x\left(\tau_k\right)\right)\right|\leq M\,\left\|\rho\right\|_\infty\int\limits_0^{R(n)}\frac{d\tau}{\tau^{1-\alpha}}\leq M\,\left\|\rho\right\|_\infty\left(R\left(n\right)\right)^\alpha.$$

Taking into account the inequality

$$\int\limits_L\left|\frac{\partial\Phi\left(x\left(\tau_k\right),\,y\right)}{\partial\vec{n}\left(y\right)}\right|dL_y\leq M\int\limits_L\frac{dL_y}{\left|x\left(\tau_k\right)-y\right|^{1-\alpha}}\leq M,\ \ \forall k\in\left\{1,2,...,n\right\},$$

and Lemma 2.1, we find

$$\left|\delta_2^n\left(x\left(\tau_k\right)\right)\right|\leq M\,\omega\left(\rho,R\left(n\right)\right)\sum_{\substack{j=1\\ j\neq k}}^{n}\int\limits_{L_j}\left|\frac{\partial\Phi\left(x\left(\tau_k\right),\,x\left(\tau_j\right)\right)}{\partial\vec{n}\left(x\left(\tau_j\right)\right)}\right|dL_y\leq$$

$$\leq M\,\omega\left(\rho,R\left(n\right)\right)\int\limits_L\left|\frac{\partial\Phi\left(x\left(\tau_k\right),\,y\right)}{\partial\vec{n}\left(y\right)}\right|dL_y\leq M\omega\left(\rho,R\left(n\right)\right).$$

Let $y\in L_j$ and $j\neq k$. It is clear that

$$\frac{\partial\Phi\left(x\left(\tau_k\right),\,y\right)}{\partial\vec{n}\left(y\right)}-\frac{\partial\Phi\left(x\left(\tau_k\right),\,x\left(\tau_j\right)\right)}{\partial\vec{n}\left(x\left(\tau_j\right)\right)}=$$

$$=\frac{\left(\overrightarrow{yx\left(\tau_k\right)},\vec{n}\left(y\right)\right)\left(\left|x\left(\tau_k\right)-x\left(\tau_j\right)\right|^2-\left|x\left(\tau_k\right)-y\right|^2\right)}{2\pi\left|x\left(\tau_k\right)-y\right|^2\left|x\left(\tau_k\right)-x\left(\tau_j\right)\right|^2}+$$

$$+\frac{\left(\overrightarrow{yx\left(\tau_j\right)},\vec{n}\left(y\right)\right)}{2\pi\left|x\left(\tau_k\right)-x\left(\tau_j\right)\right|^2}+\frac{\left(\overrightarrow{x\left(\tau_j\right)x\left(\tau_k\right)},\vec{n}\left(y\right)-\vec{n}\left(x\left(\tau_j\right)\right)\right)}{2\pi\left|x\left(\tau_k\right)-x\left(\tau_j\right)\right|^2}.$$

Then from Lemma 2.1 we obtain

$$\left|\frac{\partial\Phi\left(x\left(\tau_k\right),\,y\right)}{\partial\vec{n}\left(y\right)}-\frac{\partial\Phi\left(x\left(\tau_k\right),\,x\left(\tau_j\right)\right)}{\partial\vec{n}\left(x\left(\tau_j\right)\right)}\right|\leq$$

$$\leq M\left(\frac{R(n)}{|x(\tau_k)-y|^{2-\alpha}}+\frac{(R(n))^{1+\alpha}}{|x(\tau_k)-y|^2}+\frac{(R(n))^{\alpha}}{|x(\tau_k)-y|}\right).$$

Consequently,

$$|\delta_3^n(x(\tau_k))|\leq M\ \|\rho\|_\infty\times$$

$$\times\left(R(n)\int\limits_{r(n)}^{diamL}\frac{d\tau}{\tau^{2-\alpha}}+(R(n))^{1+\alpha}\int\limits_{r(n)}^{diamL}\frac{d\tau}{\tau^2}+(R(n))^{\alpha}\int\limits_{r(n)}^{diamL}\frac{d\tau}{\tau}\right)\leq$$

$$\leq M\ \|\rho\|_\infty(R(n))^{\alpha}\ |\ln R(n)|.$$

Besides, from the inequalities (3.2), (3.3), (4.2) and Lemma 2.1 we have

$$|\delta_4^n(x(\tau_k))|\leq M\|\rho\|_\infty(R(n))^{\alpha}\sum_{\substack{j=1\\ j\neq k}}^{n}\int\limits_{t_{j-1}}^{t_j}\left|\frac{\partial\Phi(x(\tau_k),\,x(\tau_j))}{\partial\vec{n}(x(\tau_j))}\right|dt\leq$$

$$\leq M\|\rho\|_\infty(R(n))^{\alpha}\sum_{\substack{j=1\\ j\neq k}}^{n}\int\limits_{L_j}\left|\frac{\partial\Phi(x(\tau_k),\,y)}{\partial\vec{n}(y)}\right|dL_y\leq$$

$$\leq M\|\rho\|_\infty(R(n))^{\alpha}\int\limits_{L}\left|\frac{\partial\Phi(x(\tau_k),\,y)}{\partial\vec{n}(y)}\right|dL_y\leq M\|\rho\|_\infty(R(n))^{\alpha}.$$

Summing up the estimates obtained above for $\delta_1^n(x(\tau_k))$, $\delta_2^n(x(\tau_k))$, $\delta_3^n(x(\tau_k))$, $\delta_4^n(x(\tau_k))$ and taking into account the relation $R(n)\sim\frac{1}{n}$, we finish the proof of Theorem 4.1. □

Similarly, we can prove the following theorem:

Theorem 4.2. *The expression*

$$Q_n(x(\tau_k))=\frac{b-a}{n}\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{\partial\Phi(x(\tau_k),\,x(\tau_j))}{\partial\vec{n}(x(\tau_k))}\sqrt{(x_1'(\tau_j))^2+(x_2'(\tau_j))^2}\,\rho(x(\tau_j))$$

is a quadrature formula for the normal derivative of simple layer logarithmic potential

$$Q(x)=\frac{\partial P(x)}{\partial\vec{n}(x)}=\int_L\frac{\partial\Phi(x,\,y)}{\partial\vec{n}(x)}\rho(y)dL_y,\quad x\in L,$$

at the control points $x(\tau_k)$, $k=\overline{1,\,n}$, *and the following estimate is true:*

$$\max_{k=\overline{1,n}}|Q(x(\tau_k))-Q_n(x(\tau_k))|\leq M\left(\omega(\rho,\,1/n)+\|\rho\|_\infty\frac{\ln n}{n^{\alpha}}\right).$$

Acknowledgements

This work was supported by the "University Grant " of the ASOIU (grant No. ADNSU -2018-1-01)

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Received: January 15, 2019; Accepted: April 8, 2019